PERIOD-IV

TOPIC

14 Relations and Functions, Mappings, Ratio, Proportion and Variation

14.1. RELATIONS

In this section, we shall discuss the concept of relation in detail.

Relation from a set A to set B: Let A and B be two non-empty sets. Then, a set R is said to be a relation from set A to set B if R is a subset of A × B *i.e.,* if $R \subseteq A \times B$.

Consider the two sets A = {2, 3} and B = {6. 9, 12}. The Cartesian product of A and B has 6 ordered sets which can be listed as

$$A \times B = \{(2, 6), (2, 9), (2, 12), (3, 6), (3, 9). (3, 12)\}$$

We can now obtain a subset of A × B by introducing a relation R between the first element x and the second element y of each ordered pair (x, y) as

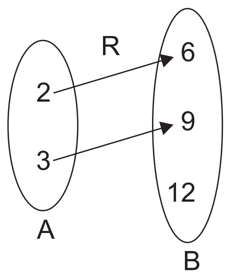
$$R = \{(x, y) : x \in A,\ y \in B,\ y \in 3x\},$$

This representation is called set-builder form of the relation R.

Also, we can write R = {(2, 6), (3, 9)}

Thus representation is called *roster form* of the relation R.

The relation R can be shown with the help of the following visual representation:



This representation is called *arrow diagram* of the relation R.

Remarks:

1. Let R be a relation from a non-empty set A to non-empty set B.
   If $(a, b) \in R$, then we say that 'a R b' *i.e.* 'a is related to b by relation R'
   If $(a, b) \notin R$, then we say that '$a \not R\ b$' *i.e.* 'a is not related to b by relation R'

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2. Let number of elements in set A = p and let number of elements in set B = q.

   Then, number of elements in set A × B = pq.

   Since, each relation from A to B is a subset of A × B.

   ∴ Number of relations from A to B = Number of subsets of A × B = 2^{pq}.

   Number of relations from A to B = 2^{pq}

3. Let A be any non-empty set. Then, a set R is said to be a relation on set A if R is a subset of A × A, *i.e.,* if R ⊆ A × A

Example 1. *To R = {(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)} a relation from set A to set B where A = {1, 2, 3, 4} and B = {1, 5, 9, 11, 15, 16}? Justify your answer.*

Solution. Given A = {1, 2, 3, 4}, B = {1, 5, 9, 11, 15, 16}

Since R = {(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)} ⊆ A × B

So, R is relation from A to B.

Function or Mapping

Let A and B be two non-empty sets. A relation f from A to B *i.e.,* a subset of A × B is called a function or mapping or a map from A to B, if

(i) For each $a \in$ A there exists $b \in$ B such that $(a, b) \in f$.

(ii) $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$

Thus, a non-void subset of A × B is a function from A to B if each element of A appears in same ordered pair in f and no two ordered pairs in f have the same first element.

If $(a, b) \in f$ then b is called the image of a under f.

14.2. KINDS OF FUNCTIONS

If A → B is a function, then f associates all elements of set A to elements in set B such that an element of set A is associated to a unique element of set B. Following these two conditions we may associate different elements of set A to different elements of set B or more than one element on set A may be associated to the same element of set B. Similarly, there may be some elements in B which do not have their pre-images in A or all elements in B may have their pre-images in A. Corresponding to each of these possibilities we define a type of a function as given below:

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One-One Function (Injection)

Definition: A function $f : A \to B$ is said to be a one-one function or an injection that it different elements of A have different images in B.

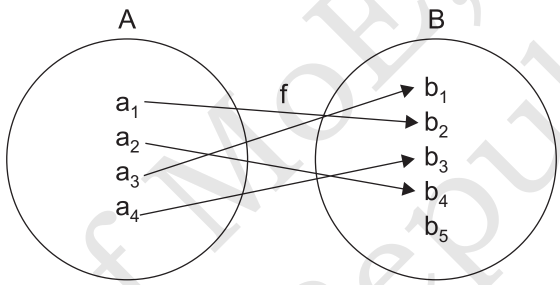
Thus, $f : A \to B$ is one-one

$$\Rightarrow \quad a \neq b \;\Rightarrow\; f(a) = f(b) \text{ for all } a, b \in A$$

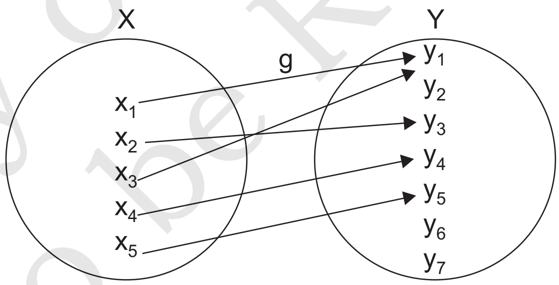
$$\Rightarrow \quad f(a) = b \;\Rightarrow\; a = b \text{ for all } a, b \in A$$

Illustration 1. *A function which associates to each country in the world, its capital is one-one because different countries have their different capitals.*

Illustration 2. *Let $f : A \to B$ and $g : X \to Y$ be two functions represented by the following diagrams:*



(a)



(b)

Let $f : A \to B$ is a one-one function. But $g : X \to Y$ is not one-one because two distinct elements x_1 and x_3 have the same image under function g.

Many-One Function

Definition: *A function $f : A \to B$ is said to be a many-one function if two or more elements of set A have the same image on B.*

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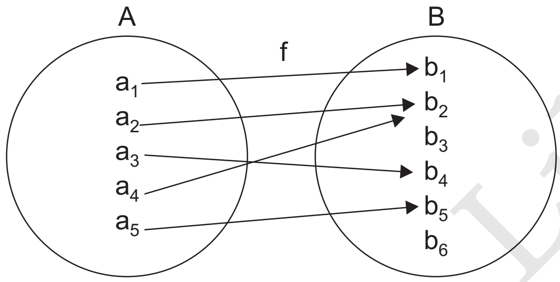
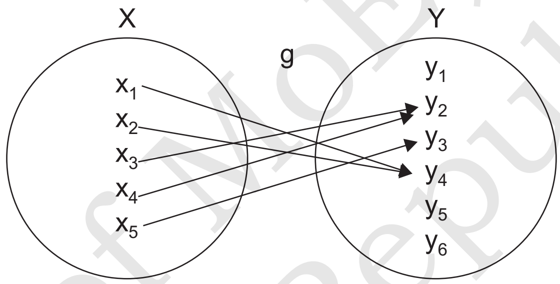
Thus, $A \to B$ *is* a many-one function if there exist $x, y \in$ A such that $x = y$ but $f(x) - f(y)$.

In other words $f : A \to B$ is many-one function if it is not a one-one function.



(a)



(b)

Illustration 2. *Let* $A - \{-1, 1, -2, 2\}$ *and* $B = \{1, 4, 9, 16\}$*. Consider* $f : A \to B$ *given by* $f(x) = x^2$*.*

Then $f(-1)$. Thus 1 and – 1 have the same image. Similarly, 2 and – 2 also have the same image. So, f is a many-one function.

Illustration 3. *Consider a function* $f : Z \to Z$ *given by* $f(x) = |x|$ *for all* $x \in Z$*. Then f is a many-one function because for every* $a \in Z$, $a \neq 0$*, we have*

$$a \neq -a \text{ but } |a| = |-a| \Rightarrow f(a) = f(-a)$$

Illustration 4. *Show that the function* $f : Z \to Z$ *given by* $f(x) = (x)^2 + x$ *for all* $x \in Z$ *is a many one function.*

Solution. Let $x, y \in$ Z. Then,

$$f(x) = f(y)$$

$$\Rightarrow \quad x^2 + x = y^2 + y$$

$$\Rightarrow \quad (x^2 - y^2) + (x - y) = 0 \quad \Rightarrow \quad (x - y)(x - y + 1) = 0$$

$$\Rightarrow \quad x = y \text{ or } y = -x - 1$$

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Since $f(x) = f(y)$ does not provide the unique solution $x = y$ but it also provides $y = x - 1$. This means that $x = y$ but $f(x) = f(y)$. When $y = - x - 1$. For example, if we put $x = 1$ in $y = - x - 1$ we obtain $y = - 2$. This shows that 1 and – 2 have the same linage under *f*. Hence, *f* is a many-one solution.

14.3. ONTO FUNCTION (SURJECTION)

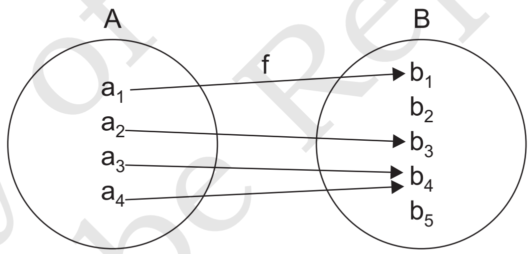
Definition: *A function $f : A \rightarrow B$* is said to be an onto fuction or a surjection if every element of B is the *f*-image of some element of Δ *i.e.*,of *f*(A) or range of *f* is the co-domain of *f*.

Thus $f : A \rightarrow B$ is a surjection if for each $b \in B$ there exists $a \in A$ such that $f(a) = b$.

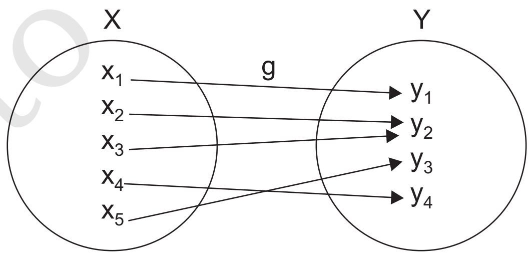
INTO FUNCTION: A function $f : A \rightarrow B$ is a into function if there exists an element in B having no pre-image in A.

In other words, $f : A \rightarrow B$ is an into function it it is not an onto function.

Illustration 1. *Let $f : A \rightarrow B$ is an into function, if it is not an onto function.*



(a)



(b)

Clearly, b_2 and b_3 are two elements in B which do not have their pre-images in A. So, $f : A \rightarrow B$ is an into functions.

Then, *f is onto because* $f(A) = \{f(- 1), f(1, f(2), f(- 2)\} = 0$, A = B.

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14.5. BIJECTION (ONE-ONE ONTO FUNCTION)

Definition: A function $f: A \to B$ is a bijection if it is one-one as well as onto.

In other words, a function $f: A \to B$ is a bijection, if it is

(*i*) one-one *i.e.* $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$.

(*ii*) onto *i.e.* for all $y \in B$, there exists is such that $f(x) = y$.

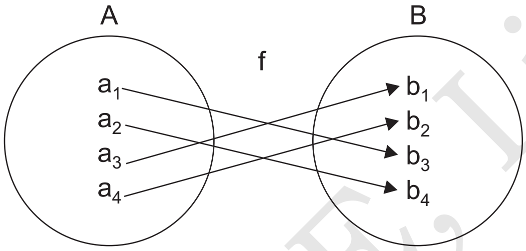

Illustration 1. *Let f : A → B be a function represented by the following diagram.*

Clearly, f is a bijection since it is both objective as well as surjective.

Illustration 2. *Prove that the function f : Q → Q given by f(x) = 2x – 3 for all x ∈ Q is a bijection.*

Solution. Let x, y be two arbitrary elements in Q. Then,

$$f(x) = f(y) \Rightarrow 2x - 3 = 2y - 3 \Rightarrow 2x = 2y \Rightarrow x = y$$

$$f(x) = f(y) \Rightarrow x = y \text{ for all } x, y \in Q.$$

So, f is an injective map.

Surjectivity: Let y be an arbitrary element of Q. Then

$$f(x) = y \Rightarrow 2x - 3 = y \Rightarrow x = \frac{y-3}{2}$$

Clearly, for all $y \in Q$, $x = \frac{y+3}{2} \in Q$. Thus, for all $y \in Q$ (co-domain) there exists $x \in Q$ (domain) given by $x = \frac{y+3}{2}$ such that $f(x) = f\left(\frac{y+3}{2}\right) = 2\left(\frac{y+3}{2}\right) - 3 = y$. That is every element in the co-domain has its pre-image in x.

So, f is a surjection.

Hence, $f: Q \to Q$ is a bijection.

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Ratio and Proportion

1. **Ratio:** The ratio of two quantitites a and b in the same units is the fraction $\frac{a}{b}$ and we write it $a : b$. In the ratio $a : b$, we call a as the first term or antecedent and b, the second term or consequence.

   Ex. The ratio 5 : 9 represents $\frac{5}{9}$ with antecedent = 4, consequent = 9

   Rule: The multiplication or division of each term of a ratio by the same non-zero number does affect the ratio.

   Ex. 4 : 5 = 8 : 10 = 12 : 15 etc. Also, 4 : 6 = 2 : 3

2. **Proportion:** The equality of two ratios is called proportion.

   If $a : b = c : d$, we write, $a : b : c : d$ and we say that a, b, c, d are in proportion.

   Here a and d are called extremes, while b and c are called mean terms.

   Produt of means = Product of extremes

   Thus, $a : b :: c : d \Rightarrow (b \times c) = (a \times d)$

3. **(*i*) Fourth proportional:** If $a : b = c : d$ then d is called the fourth proportional to a, b, c.

   (*ii*) Third proportional: If $a : b = c : d$ then c is called the third proportional to a, b, c.

   (*iii*) Mean proportional: Mean proportional between a and b is $\sqrt{ab}$

4. **(*i*) Comparison of Ratios:** We say that $(a : b) > (c : d) \Leftrightarrow \frac{a}{b} > \frac{c}{d}$

   (*ii*) Compounded Ratio: The Compounded ratio of the ratios $(a : b)$, $(c : d)$, $(e : f)$ is $(ace : bdf)$.

5. (*i*) Dublicate ratio of $(a : b)$ is $(a^2 : b^2)$

   (*ii*) Sub-dublicate ratio of $(a : b)$ is $(\sqrt{a} : \sqrt{b})$

   (*iii*) Triplicate ratio of $(a : b)$ is $(a^3 : b^3)$

   (*iv*) Sub-triplicate ratio of $(a : b)$ is $(a^{1/3} : b^{1/3})$

   (*v*) If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$, (componendo and dividendo)

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6. Variation

(*i*) We say that x is directly proportional to y, if $x = ky$ for some constant k and we write, $x \propto ky$.

(*ii*) We say that x is inversely proportional to y, if $xy = k$ for some constant k and we write, $x \propto ky$.

Example 2. *If 10% of x is equal to 20% of y, then find x : y.*

Solution. 10% of x = 20% of y

$$\Rightarrow \quad \frac{10}{100}x = \frac{20}{100}y \Rightarrow \frac{x}{10} = \frac{y}{5} \Rightarrow \frac{x}{y} = \frac{10}{5} = \frac{2}{1}$$

Hence, $x : y = 2 : 1$.

Example 3. *A man spends ₹ 500 in buying 12 tables and chairs. The cose of one table is ₹ 50 and that of one chair is ₹ 40. What is the ratio of the numbers of the chairs and tables purchased?*

Solution. Let the number of tables purchased be x.

Then, number of chairs purchased be x.

$$\therefore \quad 50x + 40(12 - x) = 500$$

$$\Leftrightarrow \quad 50x - 40x = 500 - 480$$

$$\Leftrightarrow \quad 10x = 20$$

$$\Leftrightarrow \quad x = 2$$

So, number of tables = 2 and number of chairs = 10

Hence, requried ratio = 10 : 2 = 5 : 1.

Problems on Variation

In math variation we solved numerous types of problems on variation by using different types of variation like direct variation, inverse variation and joint variation. The problems on variation are mainly related to the questions based on word problems of constant variation, word problems of direct variation, word problems on inverse variation and also word problems of joint variation. Each word problems on variation are explained step by step so that students can understand the question and their solution easily.

Example 4. *The area of an umbrealla varies directly as the square of its radius. If the radius at the umbrella is doubled, how much will be the area of the umbrella?*

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Solution. If the area of the umbrella is C and radius is R then $C = R^2$ or $C = KR^2$ where K is the constant of variation.

So, the area of the umbrella is KR^2.

Now if the radius is doubled the area will be

$$K(2R)^2 = 4KR^2 = 4C$$

So, the area will be by 4 times of normal the area of the umbrella.

Example 5. *The volume of a globe varies as the cube of its radius. Three solid globes of diameters 1½, 2 and 2½ metres are melted and formed into a new solid globe. Find the diameter of the new globe.*

Solution. $V \propto R^3$

Therefore $V = kR^3$ ...(1) [k = constant of variation]

If V_1, V_2 and V_3 cubic metres be the respective volumes of globes having radii ¾, 1 and 5/4 metres then using (1) we get,

$$V_1 = k.(3/4)^3 = 27k/64;$$
$$V_2 = k.1^3 = k;$$
$$V_3 = k.(5/4)^3 = 125k/64$$

Let V cubic metre be the volume of the new solid globe. Then,

$$V = V_1 + V_2 + V_3$$

or $V = 27k/64 + k + 125k/64$

or $V = 216k/64$

or $V = 27k/8$

Example 6. *If the radius of the new solid globe be r metre, then using (1) we get,*

Solution. $V = kr_3$

or $kr^3 = 27k/8$ or $r^3 = (3/2)^3$

or $r = 3/2$

Therefore, the diameter of the new globe = $2r = 2 \cdot 3/2 = 3$ metres.

EXERCISE

1. Let A = $\{x, y, z\}$ and R = $\{1, 2\}$. Find the number of relations from A to B.
2. If $a : 5 = b : 7 = c : 8$, then $\dfrac{a+b+c}{a}$?
3. If $p : q = r : s = t : u = 2 : 3$, then find $(mp + nr + ot) : (mq + ns + ot)$.
4. In X is in indirect variation with square of Y and when X is 3, Y is 4. What is the value of X when Y is 4?

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